Discrete Mathematics 67 (1987) 211–214
North-Holland

NOTE

TRANSITIVE STEINER TRIPLE SYSTEMS OF ORDER 25

Vladimar D. TONCHEV
Institute of Mathematics, Sofia 1090, Bulgaria

Received 8 August 1984
Revised 20 June 1986

It is shown that there exist exactly three non-isomorphic non-cyclic Steiner triple systems of order 25 possessing transitive automorphism groups. Together with the 12 non-isomorphic cyclic systems this gives a total of 15 transitive Steiner triple systems of order 25.

1. Introduction

Let us acknowledge at the outset the inspiration by R.A. Mathon, K.T. Phelps, and A. Rosa's recent survey on small Steiner triple systems [8].

A Steiner triple system of order v (STS(v), or STS) is a pair (V, B), where V is a finite set of v elements called points and B is a collection of 3-subsets of V called triples, such that any 2-subset of V is contained in exactly one triple of B. Two STS's (V_1, B_1) and (V_2, B_2) are isomorphic if there exist a bijection $f: V_1 \to V_2$ such that the induced mapping $\bar{f}: B_1 \to B_2$ is also a bijection. An automorphism of an STS is an isomorphism of the STS with itself. The set of all automorphisms forms a group under the usual composition of mappings, called the automorphism group Aut STS of the STS. An STS(v) is cyclic if it has an automorphism consisting of a single cycle of length v, and more generally transitive, if its automorphism group acts transitively on the set of points V.

The number $N(v)$ of pairwise non-isomorphic STS(v)'s is known only for $v \leqslant 15$: $N(v) = 1$ for $v = 3, 7, 9$; $N(13) = 2$; $N(15) = 80$. It is also known that $N(19) \geqslant 2\,395\,687$ [9], $N(21) \geqslant 2\,160\,980$, $N(25) \geqslant 10^{14}$; and in general, $N(v) = v^{v^2(\frac{1}{6}+o(1))}$ (see, e.g. [8] for references). On the other hand, the number $N_{tr}(v)$ of non-isomorphic transitive STS(v)'s is known only for $v \leqslant 21$: $N_{tr}(v) = 1$ for $v \leqslant 13$, $N_{tr}(15) = 2$, $N_{tr}(19) = 4$, and $N_{tr}(21) = 10$ [8]. There are exactly 12 non-isomorphic cyclic STS(25)'s [1], [3], hence $N_{tr}(25) \geqslant 12$. Our goal in this paper is to show that $N_{tr}(25) = 15$.

2. Transitive STS(25)'s

We begin with a simple but useful observation.

Proposition 2.1. *An automorphism of order* 5 *of an* STS(25) = (V, B) *does not fix any point of* V.

0012-365X/87/$3.50

Proof. Suppose that there is an automorphism f of order 5 fixing some, and therefore at least 5 points of V. Clearly f must fix the triple containing a pair of fixed points, hence the set V^* of all fixed points will define an STS of order $|V^*| \leq \frac{1}{2}(v-1)$. But this is impossible since there are no STS(v)'s for $v = 5$ or 10. □

Corollary 2.2. *If* $G = \text{Aut STS}(25)$, *then* 5^3 *does not divide the order of* G.

Proof. By Proposition 2.1 if $H < G$ and $|H| = 5^m$, then the length of each H-orbit of triples must be 5^m and since the total number of triples in an STS(25) is $100 < 5^3$, we have $m \leq 2$.

Since the cyclic STS(25)'s are known [1], [3], it follows from the above that to obtain all transitive STS(25)'s it suffices to enumerate the STS(25)'s admitting $Z_5 \times Z_5$ as a regular automorphism group.

3. The STS(25)'s admitting $Z_5 \times Z_5$

Let $V = \{(i, j): 0 \leq i, j \leq 4\}$. The set V can be considered as a 2-dimensional vector space over GF(5), and the regular action of $Z_5 \times Z_5$ on V coincides with that of the group of translations. The family of all 3-subsets of V is divided into 92 orbits under $Z_5 \times Z_5$. Using a method similar to that from [7], it can be shown by computer that exactly 480 quadruples of orbits form STS(25)'s. Any automorphism of the group $Z_5 \times Z_5$, i.e., any non-singular linear transformation of V considered as a vector space, maps an STS(25) invariant under $Z_5 \times Z_5$ to another STS(25) invariant under $Z_5 \times Z_5$ and isomorphic to the initial one. The set of all 480 solutions of STS(25)'s is partitioned into three classes under the action of the linear group GL(2, 5), each consisting of 160 STS's. Base blocks of representatives of these three classes are listed in Table 1.

Table 1

Solution	Base blocks
1	$\{(0,0), (0,1), (1,0)\}$, $\{(0,0), (0,2), (2,0)\}$, $\{(0,0), (1,1), (2,4)\}$, $\{(0,0), (1,2), (3,3)\}$,
2	$\{(0,0), (0,1), (1,0)\}$, $\{(0,0), (0,2), (2,1)\}$, $\{(0,0), (1,1), (2,3)\}$, $\{(0,0), (1,3), (3,0)\}$.
3	$\{(0,0), (0,1), (1,0)\}$, $\{(0,0), (0,2), (2,1)\}$, $\{(0,0), (1,1), (2,3)\}$, $\{(0,0), (1,3), (3,3)\}$.

The three STS's from Table 1 are pairwise non-isomorphic. For, let us consider the triples of an STS as vertices of a graph, where two triples are called adjacent if they have a common point. If P is a vertex and i an integer, let m_i be the number of unordered pairs of vertices Q, R different from P and such that exactly i other vertices are simultaneously adjacent to P, Q, R. Evidently, the characteristics $(m_0, m_1, \ldots)$ are invariant under isomorphism, i.e., any isomorphism between two STS's maps triples to triples having the same characteristics. It is perhaps worth noticing that these characteristics distinguish the 80 STS(15)'s.

The characteristics $(m_0, m_1, \ldots, m_{11})$ for the triples of the STS(25)'s No. 1, 2, 3 are listed in Table 2.

Table 2

Solution	$(m_0, m_1, \ldots, m_{11})$	Number of triples
1	(157, 489, 510, 1318, 1731, 456, 25, 0, 0, 114, 36, 15)	25
	(167, 468, 507, 1371, 1662, 492, 19, 0, 0, 102, 60, 3)	25
	(163, 501, 459, 1360, 1707, 486, 10, 0, 0, 108, 48, 9)	25
	(149, 486, 534, 1339, 1680, 462, 36, 0, 0, 108, 48, 9)	25
2	(150, 526, 449, 1353, 1734, 446, 28, 0, 0, 107, 50, 8)	75
	(159, 489, 498, 1329, 1743, 435, 33, 0, 0, 11, 42, 12)	25
3	(170, 509, 424, 1347, 1780, 437, 19, 0, 0, 99, 66, 0)	75
	(162, 519, 438, 1329, 1770, 453, 15, 0, 0, 99, 66, 0)	25

It follows by Table 2 that the solutions 1, 2, 3 are non-isomorphic. We summarize the obtained results in the following

Theorem 3.1. *Up to isomorphism, there are precisely* 3 *transitive non-cyclic Steiner triple systems of order* 25.

Combined with the results of [1], [3], this implies

Theorem 3.2. *There are precisely* 15 *isomorphism classes of transitive Steiner triple systems of order* 25.

Remarks. (i) In Hall's book [6] a solution of an STS(25) admitting $Z_5 \times Z_5$ is listed (No. 60, second solution in [6, Table 1]). This is in fact a special case of a general construction due to Bose [2]. It is straightforward to check that the transformation

$$T = \begin{bmatrix} 4 & 2 \\ 3 & 0 \end{bmatrix}$$

maps the Bose solution into our solution 1.

(ii) The transformation

$$X=\begin{bmatrix} 4 & 4 \\ 1 & 0 \end{bmatrix}$$

induces an automorphism of order 3 of the STS No. 1. Similarly, the transformation

$$Y=\begin{bmatrix} 3 & 3 \\ 4 & 1 \end{bmatrix}$$

realizes an automorphism of order 3 of the STS No. 2 and 3.

Using an algorithm from [4], we checked by computer that the full automorphism groups of the three non-cyclic STS(25)'s are in fact all of order 75.

References

[1] S. Bays, Sur les systémes cycliques de triples des Steiner, Ann. Sci. Ecole Norm. Sup. (3) 40 (1923) 55–96.
[2] R.C. Bose, On the construction of balanced incomplete block designs, Ann. Eugenics 9 (1939) 353–399.
[3] M.J. Colbourn and R.A. Mathon, On cyclic Steiner 2-designs, Ann. Discrete Math. 7 (1980) 215–253.
[4] P.B. Gibbons, R.A. Mathon and D.G. Corneil, Computing techniques for the construction and analysis of block designs, Utilitas Math. 11 (1977) 161–192.
[5] M. Hall Jr., The Theory of Groups (Macmillan, New York, 1959).
[6] M. Hall Jr., Combinatorial Theory (Blaisdell, Waltham, 1967).
[7] E.S. Kramer and D.M. Mesner, T-designs on hypergraphs, Discrete Math. 15 (1976) 263–296.
[8] R.A. Mathon, K.T. Phelps and A. Rosa, Small Steiner triple systems and their properties, Ars Combin. 15 (1983) 3–110.
[9] D.R. Stinson and H. Ferch, 2 000 000 Steiner triple systems of order 19, Math. Comput. 44 (1985), 533–535.